and columns in the matrix.

10) symmetrical matrix :

A square matrix 'A' is said to be "symmetrical matrix", if $A^T = A$ that matrix is called symmetrical matrix

11) skew symmetrical matrix :

A square matrix 'A' is said to be "skew symmetrical matrix", if $A^T = -A$ then that matrix is called "skew symmetrical matrix".

12) Nillpotent matrix :

A square matrix 'A' is said to be "Nill potent matrix", if ∃ a positive integer 'n' such that $A^n = 0$.

If 'n' is the least positive integer then 'n' is said to be "Index of A"

13) Idempotent matrix :

A matrix 'A' is said to be "Idempotent matrix", if $A^2 = A$

14) Involutary matrix :

A matrix 'A' is said to be "Involutary matrix", If $A^2 = i$

15) Equal matrix :

The two matrix 'A' and 'B' are said to be "equal matrix", if order of 'A' equal to order of 'B', and corresponding elements in 'A' and 'B' are equal.

Ex : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$

order = 2 x 2

16) Addition of matrix :

The matrices 'A' and 'B' are added, if order of 'A and 'B' are equal.

17) Multiplication of matrix :

Two matrices are said to be conformable for multiplication, if number of columns in the first matrix equal to number of row in the second matrix.

Echelon form:

echelon form of the matrix has the following propertie

1) zero row must follow a non zero row
2) The first element in a non zero row (first row) must be one
3) The zero row before a non-zero row having zeroes in increasi order.

1) Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelon form and hence find the rank of the matrix.

sol: Given matrix is $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \quad R_4 \to R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad R_3 \to R_3 - R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad R_3 \longleftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of the matrix = Number of Non-zero rows in echelon form = 3 ie; $\rho(A) = 3$.

$C_3 \to 2C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1/4,\ R_2 \to R_2/2$

$$\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix are in the form

$$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

3) **Sol:** Given matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

We know that $A = IA$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1 - R_3$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} A$$

$R_2 \to 3R_2 - 2R_1$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix} A$$

If $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ verify cayley's hamilton theorem

hence find A^{-1}

Sol: Given matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

We know that the characteristic equation of A is $|A - \lambda I|$

$$\Rightarrow \left| \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1-\lambda\left[(1-\lambda)(2-\lambda) - 1\right] + 1\left[0-2\right] + 0 = 0$$

$$\Rightarrow 1-\lambda\left[2-\lambda-2\lambda+\lambda^2-1\right] - 2 = 0$$

$$\Rightarrow 1-\lambda\left[\lambda^2-3\lambda+1\right] - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 - \lambda^3 + 3\lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow \lambda^3 + 4\lambda^2 - 4\lambda - 1 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0 \rightarrow (1)$$

Equ (1) is the characteristic equation of matrix A.

To verify cayley hamilton theorem put $\lambda = A$ in equ (1)

$$\Rightarrow A^3 - 4A^2 + 4A + 1 = 0 \rightarrow (2)$$

To find A^2:

$\therefore A^2 = A \cdot A$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1-0+0 & -1-1+0 & 0-1+0 \\ 0+0+2 & 0+1+1 & 0+1+2 \\ 2+0+4 & -2+1+2 & 0+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

To find A^3:

$\therefore A^3 = A^2 \cdot A$

$$A^3 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1\times1-2\times0-1\times2 & 1\times-1-2\times1-1\times1 & 1\times0-2\times1-1\times2 \\ 2\times1+2\times0+3\times2 & 2\times-1+2\times1+3\times1 & 2\times0+2\times1+3\times2 \\ 6\times1+1\times0+5\times2 & 6\times-1+1\times1+5\times1 & 6\times0+1\times1+5\times2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1-2 & -1-2-1 & -2-2 \\ 2+6 & -2+2+3 & 2+6 \\ 6+10 & -6+1+5 & 1+10 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

equ (2) $\rightarrow$ $$\left\{ \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4 \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + 4 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - \begin{bmatrix} 4 & -8 & -4 \\ 8 & 8 & 12 \\ 24 & 4 & 20 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = 0$$

$$\Rightarrow \begin{bmatrix} -1-4+4+1 & -4+8-4+0 & -4+4+0+0 \\ 8-8+0+0 & 3-8+4+1 & 8-12+4+0 \\ 16-24+8+0 & 0-4+4+0 & 11-20+8+1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ cayley hamilton theorem is verified.

The given matrix A satisfies its characteristic equation.

The given matrix satisfies cayley hamilton theorem

To find A Inverse :

operate A^{-1} on equ (2)

$\Rightarrow A^{-1} \cdot A^3 - 4A^2A^{-1} + 4AA^{-1} + 1 = 0$

$\Rightarrow A^2 - 4A + 4I + A^{-1} = 0$

$\Rightarrow$ The given set is linearly independent. which is contradiction to the given condition 's' is linearly dependent.

<u>If $k \neq 1$</u>

$\Rightarrow 2 \le k \le n$

from (1) $\Rightarrow a_k \alpha_k = (-a_1)\alpha_1 + (-a_2)\alpha_2 + \dots + (-a_{k+1})\alpha_{k+1} +$

$\Rightarrow \alpha_k = \left(\frac{-a_1}{a_k}\right)\alpha_1 + \left(\frac{-a_2}{a_k}\right)\alpha_2 + \dots + \left(\frac{-a_{k+1}}{a_k}\right)\alpha_{k+1}$

$\therefore \alpha_k$ can be expressed as a linear combination of its preciding vectors

Hence the Necessary Condition.

<u>Part II</u>: <u>sufficient condition</u>

If sum vector $\alpha_k \in S$, $2 \le k \le n$ can be expressed as linear combination of its preciding vectors then 's' is L.D.

Given, $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vector.

$\Rightarrow \exists$ a $b_1, b_2 \dots b_k \in F$

$\alpha_k = b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1} + (-1)\alpha_k = \bar{0}$

$\Rightarrow \{\alpha_1, \alpha_2, \dots \alpha_{k-1}, \alpha_k\}$ is linearly dependent. (::

$\Rightarrow \{\alpha_1, \alpha_2, \dots \alpha_{k-1}, \dots \alpha_n\}$ is linearly dependent

Hence the sufficient condition.

Hence the proof

<u>Theorem 16</u>:

Let $V(F)$ be a vector space, if $\alpha_1, \alpha_2, \dots \alpha_n$ are Non-zero vectors of V then, either they are linearly independent or some α_k, $2 \le k \le n$ is a linear combination of the preciding ones $\alpha_1, \alpha_2, \dots \alpha_{k-1}$.

Proof: Write the Necessary part of the above theorem

imp

<u>Theorem 17</u>: Let $V(F)$ be a vector space and $\alpha_1, \alpha_2, \dots$ if some α_k, $2 \le k \le n$ is a linear combination of the preciding ones then $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.D.

<u>Proof</u>: Write the sufficient part of theorem (15).

To Find (i) :

If $(a,b,c,d) \in W_1 \Rightarrow b-2c+d=0$

$\Rightarrow b = 2c-d$

$\therefore (a,b,c,d) = (a, 2c-d, c, d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$\therefore$ The set $\{(1,0,0,0)\ (0,2,1,0)\ (0,-1,0,1)\}$ is linearly indep

and form the basis of W_1.

$\therefore \boxed{\dim(W_1) = 3}$

To Find (ii) :

If $(a,b,c,d) \in W_2 \Rightarrow a=d,\ b=2c$

$(a,b,c,d) = (d, 2c, c, d)$

$= c(0,2,1,0) + d(1,0,0,1)$

$\therefore$ The set $\{(0,2,1,0)\ (1,0,0,1)\}$ is linearly independent form the basis of W_2.

$\therefore \boxed{\dim(W_2) = 2}$

To Find (iii) :

If $(a,b,c,d) \in W_1 \cap W_2$

$\Rightarrow b-2c+d=0 \longrightarrow (1)$

and $a=d,\ b=2c$

put $b=2c$ in equ (1) $\Rightarrow \cancel{2c} - \cancel{2c} + d = 0$

$\Rightarrow d=0$

$\Rightarrow a=0$

$\therefore (a,b,c,d) = (0,2c,c,0)$

$= c(0,2,1,0)$

$\therefore$ The set $\{(0,2,1,0)\}$ is linearly indepent and forms the ba

of $W_1 \cap W_2$.

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

We know that $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim($

$= 3+2-1$

$= 4$

To prove $\left(\frac{V}{W}, +\right)$ is a commutative group :

i) Commutative axiom :

$\forall\ \alpha, \beta \in V \Rightarrow \alpha+\beta \in V$

If $\alpha, \beta \in V \Rightarrow W+\alpha,\ W+\beta \in \frac{V}{W}$

Then $\alpha+\beta$ $(W+\alpha)+(W+\beta) = W+(\alpha+\beta) \in \frac{V}{W}$ upto

$\therefore \frac{V}{W}$ is closed under the given composition addition

ii) Associative axiom :

$\forall\ \alpha, \beta, \gamma \in V \Rightarrow \alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$

If $\alpha, \beta, \gamma \in V \Rightarrow (W+\alpha), (W+\beta), (W+\gamma) \in \frac{V}{W}$

consider,

$$(W+\alpha)+[(W+\beta)+(W+\gamma)] = (W+\alpha)+[W+(\beta+\gamma)]$$
$$= W+[\alpha+(\beta+\gamma)]$$
$$= W+[(\alpha+\beta)+\gamma]$$
$$= W+[(\alpha+\beta)+(W+\gamma)]$$
$$= [W+(\alpha+\beta)]+(W+\gamma)$$

i.e., $(W+\alpha)+[(W+\beta)+(W+\gamma)] = [(W+\alpha)+(W+\beta)]+(W+\gamma)$

$\therefore$ Composition '+' is associative in $\frac{V}{W}$

iii) Identity axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $\bar{0} \in V$ such that

$a+0 = a = 0+a$.

$\alpha+\bar{0} = \alpha = \bar{0}+\alpha$

If $\alpha, \bar{0} \in V \Rightarrow W+\alpha,\ W+\bar{0} \in \frac{V}{W}$

Consider, $(W+\alpha)+(W+\bar{0}) = W+(\alpha+\bar{0})$

$= W+\alpha$

similarly, $(W+\bar{0})+(W+\alpha) = W+\alpha$.

$\therefore (W+\bar{0}) = W$ is the identity element in $\frac{V}{W}$

iv) Inverse axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $-\alpha \in V$ such that zero vector.

Since, $1 0 + 0\alpha_2 + 0\alpha_3 + \dots + 0\alpha_n = 0$

Here $1, 0, 0 \dots 0 \in F$ not all zeros

$\Rightarrow$ 'S' is linearly dependent.

Hence the proof

Theorem 12 :

Let $V(F)$ be a vector space. If two vector in V are linearly dependent then one of them is a scalar multiple of the other.

proof : Given $V(F)$ be a vector space

Let $S = \{\alpha, \beta\}$ are linearly dependent.

Let there exist scalars $a_1, a_2 \in F$ not all zero

such that $a_1\alpha + a_2\beta = 0$

Let $a_1 \neq 0$ then $a_1\alpha + a_2\beta = 0$

$\Rightarrow a_1\alpha = -a_2\beta$

$\Rightarrow \alpha = \left(\frac{a_2}{a_1}\right)\beta$

$\Rightarrow$ 'α' is the scalar multiple of β

Similarly, if $a_2 \neq 0 \Rightarrow \beta$ is the scalar multiple of α

Hence the proof

Theorem 13 :

Let $V(F)$ be a vector space then every super set of a LD set of V is linearly dependent

Proof : Given $V(F)$ be a vector space

To prove that every super set of a Linearly dependent set of V is linearly dependent

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ is linearly dependent

Let there exist $a_1, a_2 \dots a_n \in F$ not all zero

such that $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0} \rightarrow (1)$

Let $S^1 = \{\alpha_1, \alpha_2 \dots \alpha_n, \beta_1, \beta_2 \dots \beta_m\}$ be the super set of S

$\therefore$ equ (1) $\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \dots + 0\beta_m = 0$

Where scalars are $a_1, a_2 \dots a_n, 0, 0 \dots 0 \in F$ Not all zero

$\Rightarrow S^1$ is linearly dependent.

Hence the proof.

Theorem 14: Let $V(F)$ is a vector space, a subset of a linearly independent set of 'V' is linearly independent.

proof: Given $V(F)$ is a vector space

To prove that a sub set of a linearly independent set, of 'V' is L.I

let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is L.I

Let there exist scalar $a_1, a_2 \ldots a_n \in F$ such that

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$

$\Rightarrow a_1 = 0, a_2 = 0 \ldots a_n = 0$

let $S' = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ $(m < n)$

To S.T S' is L.I :

Let there exist scalars $a_1 a_2 \ldots a_m \in F$

such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = \bar{0}$

$\Rightarrow a_1 = 0, a_2 = 0, \ldots a_m = 0$

$\Rightarrow S'$ is linearly independent.

Hence the proof.

Imp

Theorem 15 :

$V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a finite sub set of Non-zero vectors of $V(F)$. Then 'S' is linearly dependent if and only iff sum vector $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vectors.

proof: Given $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a Non-zero sub set of vectors in $V(F)$.

part I :

If 'S' is linearly dependent, then S.T there exist some vectors $\alpha_k \in S$, $2 \le k \le n$ can be expressed as linear combination of its preciding vectors.

Given, 'S' is linearly dependent

let 'k' be the largest suffix such that $a_k \neq 0$

$\therefore a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_n = \bar{0}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1} + a_k\alpha_k = \bar{0} \rightarrow (1)$

If $k = 1$

Then $a_1\alpha_1 = \bar{0} \Rightarrow a_1 = 0$ $[\because \alpha_1 \neq \bar{0}]$

$\Rightarrow$ The given set is linearly independent. which is contradiction to the given condition 's' is linearly dependent

<u>If $k \neq 1$</u>

$\Rightarrow 2 \le k \le n$

From (1) $\Rightarrow a_k \alpha_k = (-a_1)\alpha_1 + (-a_2)\alpha_2 + \dots + (-a_{k+1})\alpha_{k+1}$

$\Rightarrow \alpha_k = \left(\frac{-a_1}{a_k}\right)\alpha_1 + \left(\frac{-a_2}{a_k}\right)\alpha_2 + \dots + \left(\frac{-a_{k+1}}{a_k}\right)\alpha_{k+1}$

$\therefore \alpha_k$ can be expressed as a linear combination of its preciding vectors

Hence the Neccessary Condition.

<u>Part II: sufficient condition</u>

If sum vector $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vectors then 's' is L.D.

Given, $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vector.

$\Rightarrow \exists$ a $b_1, b_2 \dots b_k \in F$

$\alpha_k = b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1} + (-1)\alpha_k = \bar{0}$

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1}, \alpha_k\}$ is linearly dependent.

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1} \dots \alpha_n\}$ is linearly dependent

Hence the sufficient condition.

Hence the proof

<u>Theorem 16</u>:

Let $V(F)$ be a vector space, if $\alpha_1, \alpha_2 \dots \alpha_n$ are Non-zero vectors of V then, either they are linearly independent or some α_k, $2 \le k \le n$ is a linear combination of the preciding ones $\alpha_1, \alpha_2 \dots \alpha_{k-1}$.

Proof: Write the Neccessary part of the above theorem.

Imp

<u>Theorem 17</u>: Let $V(F)$ be a vector space and $\alpha_1, \alpha_2 \dots \alpha_n$ if some α_k, $2 \le k \le n$ is a linear combination of the preciding ones then $\alpha_1, \alpha_2 \dots \alpha_n$ are L.D.

<u>Proof</u>: Write the sufficient part of theorem (15).

Theor

Let $V(F)$ be a vetor space and $S = \{\alpha_1, \alpha_2 \dots \alpha_n\} \subseteq V$. If some α_i, $2 \le i \le n$ is a linear combination of the preciding once $\alpha_1, \alpha_2 \dots \alpha_{i-1}$ then $L(S) = L(S')$, where $S' = \{\alpha_1, \alpha_2 \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_n\}$.

Proof: Given that $V(F)$ be a vetor space.

and $S = \{\alpha_1, \alpha_2 \dots \alpha_{i-1}, \alpha_i, \alpha_{i+1} \dots \alpha_n\}$

also given $S' = \{\alpha_1, \alpha_2 \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_n\}$

clearly $S' \subseteq S$

$\Rightarrow L(S') \subset L(S) \rightarrow (1)$

Now, To prove that $L(S) \subseteq L(S')$

let $\alpha \in L(S)$

$\Rightarrow$ 'α' can be expressed as linear combination of vectors of 'S'

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{i-1}\alpha_{i-1} + a_i\alpha_i + a_i\alpha_i + \dots + a_{i+1}\alpha_{i+1} + \dots + a_n\alpha_n$ (2)

given that α_i is the linear combination of $\alpha_1, \alpha_2 \dots \alpha_{i-1}$

$\Rightarrow \alpha_i = b_1\alpha_1 + b_2\alpha_2 + \dots + b_{i-1}\alpha_{i-1}$

From (2) $\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{i-1}\alpha_{i-1} + a_i(b_1\alpha_1 + b_2\alpha_2 + \dots + b_{i-1}\alpha_{i-1}) + a_{i+1}\alpha_{i+1} + \dots + a_n\alpha_n$

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{i-1}\alpha_{i-1} + a_ib_1\alpha_1 + a_ib_2\alpha_2 + \dots + a_ib_{i-1}\alpha_{i-1} + a_{i+1}\alpha_{i+1} + \dots + a_n\alpha_n$

$\Rightarrow \alpha = \alpha_1(a_1 + a_ib_1) + \alpha_2(a_2 + a_ib_2) + \dots + \alpha_{i-1}(a_{i-1} + a_ib_{i-1}) + a_{i+1}\alpha_{i+1} + \dots + a_n\alpha_n$

i.e, 'α' can be expressed as a linear combination of vectors of $\alpha_1, \alpha_2 \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_n$

$\Rightarrow \alpha$ = linear combination of vectors of S'.

$\Rightarrow \alpha \in L(S')$.

$\therefore$ If $\alpha \in L(S) \Rightarrow \alpha \in L(S')$.

$\Rightarrow L(S) \subseteq L(S') \rightarrow (3)$

$\therefore$ From (1) & (3) $\Rightarrow L(S') \subseteq L(S)$, $L(S) \subseteq L(S')$

$\Rightarrow L(S) = L(S')$.

Hence the proof.

Vector space - II

Define basis of a vector space :

A Non-empty sub set 's' of a vector space $V(F)$ is set to be "a basis", if i) 's' is linearly independent
ii) $L(S) = V$, ie; every vector in 'V' can be expressed as a linear combination of vectors in 's'.

Note:

1) The set $S = \{e_1, e_2 \ldots e_n\}$ be the standard basis for the vector space $V_n(F)$
2) The set $S = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ be the standard basis for the vector spac $V_3(F)$

Finite dimensional vector space :

A vector space $V(F)$ is said to "Finite dimensional" (or) "finitely generated". If there exist a finite sub set of 'V' such that $L(S) = V$.

imp

Theroem 19 :

If $V(F)$ is a finite dimensional vector space, then there exist a basis set of V.

(or)

Every finite dimensional vector space has a basis.

Proof : Given $V(F)$ is a finite dimensional vector space.

let $S = \{\alpha_1, \alpha_2 \ldots \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots \alpha_n\}$ be any Non-Empty sub set of $V(F)$.

ie, $L(S) = V$, since 'V' is a vector space.

Now, we S.T 's' is linearly independent :

If 's' is linearly independent then 's' it self is a basis of $V(F)$.

If 's' is linearlyindependent then linearly dependent We know that by one of the theorem there exist a vector α_k in 's' can be expressed as a linear combination of it's preceding vectors.

$\therefore$ There exist scalars $a_1, a_2 \ldots a_{k-1} \in F$ such that

$$\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1}$$

Remove the vector α_k from the set 's', we get a set S_1

$S_1 = \{\alpha_1, \alpha_2 \ldots \alpha_{k-1}, \alpha_{k+1} \ldots \alpha_n\}$

If 'S_1' is linearly independent then 'S_1' it self is a basis of $V(F)$

If S_1 is linearly dependent

then proced of above, after sum finite number of steps we get a set containg only one vector say $S_k = \{\alpha_1\}$

WKT, S_k is linearly independent [$\because a_1\alpha_1 = 0 \Rightarrow a_1 = 0$]

$\therefore$ 'S_k' is the basis of $V(F)$.

$\therefore$ "Every finite dimensional vector space has a basis"

Hence the proof

Problems :

S.T, the set $S = \{(1,0,-2)\ (1,2,1)\ (0,-3,2)\}$ forms a basis of a vector space $V_3(R)$.

Sol: Given, $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,0,-2)$
$\alpha_2 = (1,2,1)$
$\alpha_3 = (0,-3,2)$

To prove 's' is a basis, we S.T

i) 's' is linearly independent:

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(1,0,-2), + a_2(1,2,1) + a_3(0,-3,2) = (0,0,0)$

$\Rightarrow a_1 + a_2 + 0. \quad 2a_2 - 3a_3 = 0 \to ② \quad -2a_1 + a_2 + 2a_3 = 0 \to ③$

$\Rightarrow a_1 + a_2 = 0 \to ①$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix}$ $R_3 \Rightarrow 2R_1 + R_3$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix}$ $R_3 \Rightarrow 3R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & -13 \end{bmatrix}$$

The above matrix is an echlon form

$\therefore$ Rank of $A = \rho(A)$ = No of Non-zero rows in the echlon form = 3

and Ran No.of variable = 3

No.of variable = No of Non-zero rows

$\therefore$ The given set 's' is linearly independent

(ii) $\underline{L(S) = V_3(R)}$:

Let $(a,b,c) \in V_3(R)$ be any vector in $V_3(R)$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \longrightarrow (a)$$

$$\Rightarrow a_1(1,0,-2) + a_2(1,2,1) + a_3(0,-3,2) = (a,b,c)$$

$$\Rightarrow \therefore a_1 + a_2 = a$$

$$2a_2 - 3a_3 = b$$

$$-2a_1 + a_2 + 2a_3 = c$$

The above Equation can be written in matrix formas follows.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Argumented Matrix,
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ -2 & 1 & 2 & c \end{array}\right] \quad R_3 \to 2R_1 + R_3$$

2a+c

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ 0 & 3 & 2 & 2a+c \end{array}\right] \quad R_3 \Rightarrow 3R_2 - 2R_3$$

$3b - 4a + 2c$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 2 & -3 & b \\ 0 & 0 & -13 & 3b-4a-2c \end{array}\right]$$

The above matrix can be written in Equation form as follow

$$a_1 + a_2 + 0 \cdot a_3 = a \longrightarrow (4)$$

$$0 \cdot a_1 + 2a_2 - 3a_3 = b \longrightarrow (5)$$

$0a_1 + 0a_2 - 13a_3 = 3b - 4a - 2c$ ⑤ $a_1 = \frac{a-b+2c}{2}$, $a_2 = \frac{a-b+2c}{2}$, $a_3 = \frac{2b-a}{2}$.

$\rightarrow a_3 = -\frac{1}{13}(3b-4a-2c)$

$\Rightarrow a_3 = \frac{4a-3b+2c}{13}$

put a_3 values in equ ⑤ $\Rightarrow 2a_2 - 3\left(\frac{4a-3b+2c}{13}\right) = b$

$\Rightarrow 2a_2 = \frac{3}{13}(4a-3b+2c)+b$

$\Rightarrow 2a_2 = \frac{12a-9b+6c+13b}{13}$

$\Rightarrow a_2 = \frac{12a+4b+6c}{26}$

$\Rightarrow a_2 = \frac{2(6a+2b+3c)}{26}$

$\Rightarrow a_2 = \frac{6a+2b+3c}{13}$.

put a_2, a_3 values in equ ④ ④ $a_1 = \frac{a+c}{2}$, $a_2 = \frac{2b-a-c}{2}$, $a_3 = b-c$

$a_1 + \frac{6a+2b+3c}{13} = a$

$a_1 = \frac{-6a-2b-3c}{13} + a$

$a_1 = \frac{-6a-2b-3c+13a}{13}$

$a_1 = \frac{7a-2b-3c}{13}$.

put a_1, a_2, a_3 values in equ ⓐ

$(a,b,c) = \frac{1}{13}(7a-2b-3c)(1,0,2) + \frac{1}{13}(6a+2b+3c)(1,2,1) + \frac{1}{13}(4a-3b+2c)(0,-3,2)$

$\therefore$'s' is linearly independent and $L(S) = V$

$\therefore$ 'S' is a basis of $V_3(R)$. $a_1 = \frac{1}{9}(-2a+4b+3c)$, $a_2 = \frac{1}{9}(a-b-3c)$, $a_3 = \frac{1}{9}(a \ldots)$

S.T the set 'S' = $\{(1,2,1), (2,1,0), (1,-1,2)\}$ form a basis of a vector space $V_3(F)$.

S.T a set 'S' = $\{(2,1,4)\ (1,-1,2)\ (3,1,-2)\}$ forms a basis of vector space $V_3(F)$ $a_1 = \frac{2b+c}{c}$ $a_2 = \frac{6a-16b+c}{24}$ $a_3 = \frac{2a-c}{8}$

1) S.T the set $S = \{(1,1,1)\ (-1,1,1)\ (1,0,1)\}$ forms a basis of R^3

2) S.T the set $S = \{(2,1,0)\ (2,1,1)\ (2,2,1)\}$ forms a basis of R^3

3) S.T the set $S = \{(1,1,1,1)\ (0,1,1,1)\ (0,0,1,1)\ (0,0,0,1)\}$ forms a basis of R^3 $a_1 = a$ $a_2 = b - a$ $a_3 = c - b$ $a_4 = d - c$

4) S.T the vectors $\{(1,1,2)\ (1,2,5)\ (5,3,4)\}$ of $R^3(R)$ donot form a basis of $R^3(R)$.

Sol: Given, $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,1,2)$

$\alpha_2 = (1,2,5)$

$\alpha_3 = (5,3,4)$

To prove 's' is a basis, We S.T

i)

let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

$$\Rightarrow a_1(1,1,2) + a_2(1,2,5) + a_3(5,3,4) = (0,0,0)$$

$$\Rightarrow a_1 + a_2 + 5a_3 = 0 \rightarrow (1)$$

$$a_1 + 2a_2 + 3a_3 = 0 \rightarrow (2)$$

$$2a_1 + 5a_2 + 4a_3 = 0 \rightarrow (3)$$

The above equ's can be written in matrix

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\therefore$ co-efficient Matrix $= \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is an echlon form.

$\therefore$ Rank of $A = \rho(A) =$ No.of Non-zero rows in the echlon form $= 2$

$0\ 3\ -6$
$0\ 3\ -6$
$0\ 0\ 0$

$1\ 2\ 3$
$1\ 1\ 5$
$0\ 1\ -2$

$2\ 5\ 4$
$-2\ 2\ 10$
$0\ 3\ -6$

And No. of variable = 3

No of ~~vabia~~ variable $\neq$ No. of Non-zero rows

$\therefore$ The given set 's' is not linearly independent

$\therefore$'s' is not a basis of $R^3(R)$.

NOTE

If 's' is any linearly independent in $V(F)$. then 's' can be extended to form the basis of $V(F)$.

Dimension of a vector space :

If $V(F)$ is a vector space over the field F. then the no. of elements presenting any basis of $V(F)$ is said to be "dimension of a vector space $V(F)$", and it is denoted by $\dim V(F)$

NOTE :

$\dim V_n(F) = n$

Theorem 20 :

Let $V(F)$ be a Finite dimensional vector space then any two basis of 'V' has the same no of elements.

Proof : Given $V(F)$ be a Finite dimensional vector space over the field 'F'.

let $S_1 = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ and

$S_2 = \{\beta_1, \beta_2 \ldots \alpha_n\}$ are any two basis of $V(F)$.

Now, We show that $m = n$

Case (i): consider S_1 is the basis and S_2 is any linearly independent set $V(F)$.

If S_1 is the basis of $V(F)$.

$\Rightarrow \dim V(F) =$ The no. of vector present in $S_1 = m$

By one of the theorem, WKT the linearly independent set can be extended to form the basis of $V(F)$

$\therefore n \leq m \rightarrow (1)$

Case (ii): consider S_2 is the basis and S_1 is any linearly independent set $V(F)$.

If S_2 is the basis of $V(F)$

$\Rightarrow \dim V(F) =$ Th no. of vector present in $S_2 = n$

Again by one of the theorems S_1 can be extended to for the basis of V(F).

$$\therefore m \le n \quad \text{---} (2)$$

$\therefore$ From (1) & (2) $\Rightarrow$ ~~Imes~~ $n \le m,\ m \le n$

$$\Rightarrow m = n$$

Co-ordinates :

Let V(F) be a finite dimensional vector space and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be an order basis of V. If $\alpha \in V$ & α can be uniquely expressed as,

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n, \text{ Where } a_1, a_2, \ldots a_n \in F$$

The scalars $a_1, a_2 \ldots a_n$ are called co-ordinats of α.

The matrix $X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is called the co-ordinate matrix of 'α'.

Find the coordinates of the vectors $(2,1,-6)$ of R^3 Relative to the order basis $\{(1,1,2)\ (3,-1,0)\ (2,0,-1)\}$

Sol: Given $\alpha = (2,1,-6)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where
$$\alpha_1 = (1,1,2)$$
$$\alpha_2 = (3,-1,0)$$
$$\alpha_3 = (2,0,-1)$$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$\Rightarrow (2,1,-6) = a_1(1,1,2) + a_2(3,-1,0) + a_3(2,0,-1)$$

$$\Rightarrow (2,1,-6) = a_1 + a_2 + 2a_3,\ a_1 - a_2 + 0,\ 2a_1 + 0 - a_3$$

$$\therefore a_1 + a_2 + a_3 = 2$$
$$a_1 - a_2 = 1$$
$$2a_1 - a_3 = -6$$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is an echlcon matrix
$\therefore$ Rank of $A = \rho(A)$ = no of non-zero rows in the echelon form 3 and no.of variables = 3

$\therefore$ No of variables = No of non-zero rows

ii) $L(S) = R^3$;

Let $(a,b,c) \in R^3$ any vector in (R^3)

let their exists scalers $a_1, a_2, a_3, a_4 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = (a,b,c,d) \to ⓐ$$

$\Rightarrow a_1(1,1,1,1) + a_2(0,1,1,1) + a_3(0,0,1,1) + a_4(0,0,0,1) = (a,b,c,d)$

$a_1 = a \to ①$

$a_1 + a_2 = b \to ②$

$a_1 + a_2 + a_3 = c \to ③$

$a_1 + a_2 + a_3 + a_4 = d \to ④$

3) If W_1 and W_2 are the sub space of $V_4(R)$ generated by $\{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,1)\}$ and $\{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$ respectively, find the dimensional of W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$.

Sol: Given that $W_1 = \{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,-1)\}$.

$W_2 = \{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$

To find dimension $\dim(W_1)$.

Arranging the vectors in the first set as rows of matrix and reducing to echlon form.

$$\Rightarrow W_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\begin{array}{cccc} 1 & 2 & 3 & 0 \\ -1 & 1 & 0 & +1 \\ \hline 0 & 1 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{array}{cccc} 2 & 3 & 3 & -1 \\ -2 & -2 & 0 & +2 \\ \hline 0 & 1 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,1,0,1)\ (0,1,3,1)\}$

$\Rightarrow \boxed{\dim(W_1) = 2}$

To Find $\dim(W_2)$:

Arranging the vectors in the second set as a rows of a matrix and reducing to echlon form.

$$\Rightarrow W_2 = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{array}{cccc} 2 & 3 & 2 & -3 \\ -2 & -4 & -4 & +4 \\ \hline 0 & -1 & -2 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\begin{array}{cccc} 1 & 3 & 4 & -3 \\ -1 & -2 & -2 & +2 \\ \hline 0 & 1 & 2 & -1 \end{array}$$

3) sol: Given $s = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (2,1,4)$

$\alpha_2 = (1,-1,2)$

$\alpha_3 = (3,1,-2)$

To prove 's' is a basis, we show that

i) 's' is linearly independent:

let ∃ scalers $a_1, a_2\ a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (0,0,0)$

$2a_1 + a_2 + 3a_3 = 0 \rightarrow ①$

$a_1 - a_2 + a_3 = 0 \rightarrow ②$

$4a_1 + 2a_2 - 2a_3 = 0 \rightarrow ③$

The above equ can be written in matrix form as follows

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ co-efficient matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$ $R_2 \rightarrow 2R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

The above matrix is an echlon matrix.

∴ Rank of $A = \rho(A)$ = no of non-zero rows in the echlon from is 3 and no.of variables = 3

∴ No.of variables = no of non-zero rows in the echlon from is 3

∴ The given set 's' is linearly independent.

ii) $L(S) = V_3(F)$:

let $(a,b,c) \in V_3(F)$ be any vector in $V_3(F)$.

Let ∃ scalers $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \rightarrow ⑥$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (a,b,c)$

$\Rightarrow 2a_1 + a_2 + 3a_3 = a$

$a_1 - a_2 + a_3 = b$

$4a_1 + 2a_2 - 2a_3 = c$

The above equ's can be matrix form as follows.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$R_5 \to R_5 - R_4$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non - zero rows are $\{(1,1,0,-1)\ (0,1,3,1)\ (0,0,-2,-4)\}$

$\Rightarrow \boxed{\dim(W_1+W_2) = 3}$

To Find $\dim(W_1 \cap W_2)$:

We know that $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1+W_2)$

$= 2 + 2 - 3$

$= 4 - 3$

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

$\dim(W_1) = 2$, $\dim(W_1 \cap W_2) =$

$\dim(W_2) = 1$

$\dim(W_1+W_2) = 3$

4) If W_1 and W_2 are the sub space of $V_4(R)$ generated by $\{(1,1,-1,2)\ (2,1,3,0)\ (3,2,2,2)\}$ and $\{(1,-1,0,1)\ (-1,1,0,-1)\}$ respectively then find dim W_1, W_2, $W_1 \cap W_2$ and W_1+W_2

Imp 5) Let W_1 and W_2 be two sub spaces of R^4 given by $W_1 = \{(a,b,c,d) / b-2c+d=0\}$, $W_2 \{(a,b,c,d) / a=d, b=2c\}$ Find the basis and dimension of

i) W_1 (ii) W_2 (iii) $W_1 \cap W_2$ and hence Find $\dim(W_1+W_2)$

Sol: Given $W_1 = \{(a,b,c,d) / b-2c+d=0\}$

$W_2 = \{(a,b,c,d) / a=d, b=2c\}$ are any two subs of R^4.

put $a_3 = 1$ in equ (3) $\Rightarrow a_2 - 1 = 4$
$\Rightarrow a_2 = 5$

put a_2 in equ (2) $\Rightarrow a_1 + 5 = 3$
$\Rightarrow a_1 = 3 - 5$
$\Rightarrow a_1 = -2$

put a_1 in equ (1) $\Rightarrow -2 + a_4 = 2$
$\Rightarrow a_4 = 2 + 2$
$\Rightarrow a_4 = 4$

$\therefore$ The co-ordinate matrix of $(2, 3, 4, -1)$ are $\begin{bmatrix} -2 \\ 5 \\ 1 \\ 4 \end{bmatrix}$

1) If W is a sub space of $V_4(R)$ generated by the vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$ then find a basis of W and its dimensional

Sol: Given $W = \{(1,-2,5,-3)\ (2,3,1,-4)\ (3,8,-3,-5)\}$

Aranging the given vectors as a rows of a matrix and reducing to echlon form

$$\Rightarrow W = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3+6-15$

$0\ 14$

Here the Non-zero rows are $\{(1,-2,5,-3)\ (0,7,-9,2)\}$ is a basis of W

$\therefore \dim(W) = 2$

2) Find the basis of W and its dimention. $\dim(W) = 2$

a) $W = \{(1,2,0)\ (-1,0,1)\ (0,2,1)\}$ of $V_3(R)$

b) $W = \{(1,7,3)\ (1,-1,0)\ (1,2,1)\ (0,3,1)\}$ of R^3 $\dim(W) = 2$

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the Non-zero rows are $\{(1,2,2,-2)\ (0,-1,-2,1)\}$

$\therefore \boxed{\dim(W_2) = 2}$

To find $\dim(W_1+W_2)$.

Arranging the given vector in W_1 and W_2 as a rows of matrix and reducing to echelon form.

$$\Rightarrow W_1 + W_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$, $R_5 \to R_5 - 2R_1$
$R_6 \to R_6 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, R_4 $R_4 - R_2$
$R_5 \to R_5 - R_2$, $R_6 \to R_6 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_6$

To Find (i):

If $(a,b,c,d) \in W_1 \Rightarrow b-2c+d=0$

$\Rightarrow b = 2c-d$

$\therefore (a,b,c,d) = (a, 2c-d, c, d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$\therefore$ The set $\{(1,0,0,0)\ (0,2,1,0)\ (0,-1,0,1)\}$ is linearly indep and form the basis of W_1.

$\therefore \boxed{\dim(W_1) = 3}$

To Find (ii):

If $(a,b,c,d) \in W_2 \Rightarrow a=d,\ b=2c$

$(a,b,c,d) = (d, 2c, c, d)$

$= c(0,2,1,0) + d(1,0,0,1)$

$\therefore$ The set $\{(0,2,1,0)\ (1,0,0,1)\}$ is linearly independent form the basis of W_2.

$\therefore \boxed{\dim(W_2) = 2}$

To Find (iii).

If $(a,b,c,d) \in W_1 \cap W_2$

$\Rightarrow b-2c+d=0 \longrightarrow (1)$

and $a=d,\ b=2c$

put $b=2c$ in equ (1) $\Rightarrow \cancel{2c}-\cancel{2c}+d=0$

$\Rightarrow d=0$

$\Rightarrow a=0$

$\therefore (a,b,c,d) = (0, 2c, c, 0)$

$= c(0,2,1,0)$

$\therefore$ The set $\{(0,2,1,0)\}$ is linearly indepent and forms the ba of $W_1 \cap W_2$.

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

We know that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W$

$= 3+2-1$

$= 4$

put $b_1 = 0, b_2 = 0, \ldots b_k = 0$ in equ (1), we get

$\Rightarrow c_1\gamma_1 + c_2\gamma_2 + \ldots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = 0$

since γ's and α's are the vectors present in the basis B_1 of W_1

∵ by definition of basis linearly independent

$\Rightarrow c_1 = c_2 = \ldots = c_k = a_1 = a_2 = \ldots a_m = 0$

$c_1 = c_2 = \ldots = c_k = a_1 = a_2 = \ldots a_m = b_1 = b_2 = \ldots b_k = 0$

∴ The set S' is linearly independent.

To prove $L(S') = W_1 + W_2$:

We know that S' is a sub set of $W_1 + W_2$

∴ $L(S') \subseteq W_1 + W_2 \rightarrow$ (4)

Let $\delta \in W_1 + W_2$

$\Rightarrow \delta = \alpha + \beta$, where $\alpha \in W_1, \beta \in W_2$ (∵ Definition of linear sum)

= (linear combination of α's and γ's) + (linear combination of β's and γ's)

$\Rightarrow \delta =$ (linear combination of α's, β's and γ's) $\in L(S')$

$\Rightarrow \delta \in L(S')$

∴ If $\delta \in W_1 + W_2 \Rightarrow \delta \in L(S')$

$\Rightarrow W_1 + W_2 \subseteq L(S') \rightarrow$ (5)

∴ From (4) & (5) $\Rightarrow L(S') \subseteq W_1 + W_2$, $W_1 + W_2 \subseteq L(S')$

$\Rightarrow L(S') = W_1 + W_2$

∴ S' is a basis of $W_1 + W_2$

$\Rightarrow \dim(W_1 + W_2) = k + m + t$

∴ $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Hence the proof

3) Sol: Given $\alpha = (2,3,4,-1)$

let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Where $\alpha_1 = (1,1,1,2)$

$\alpha_2 = \cancel{(0,0,1,1)}\ (1,-1,0,0)$

$\alpha_3 = (0,0,1,1)$

$\alpha_4 = (0,1,0,0)$

let there exist scalar $a_1, a_2, a_3, a_4 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$(2,3,4,-1) = a_1(1,1,1,2) + a_2(1,-1,0,0) + a_3(0,0,1,1) + a_4(0,1,0,0)$$

$$(2,3,4,-1) = (a_1+a_2,\ a_1-a_2+a_4,\ a_1+a_3,\ 2a_1+a_3)$$

$\therefore$ $a_1 + a_2 = 2 \rightarrow (1)$

$a_1 - a_2 + a_4 = 3 \rightarrow (2)$

$a_1 + a_3 = 4 \rightarrow (3)$

$2a_1 + a_3 = -1 \rightarrow (4)$

solving (3) & (4)

$a_1 + \cancel{a_3} = 4$

$-2a_1 - a_3 = +1$

$-a_1 = 5 \Rightarrow \boxed{a_1 = -5}$

put a_1 value in equ (1) $\Rightarrow -5 + a_2 = 2$

$\Rightarrow a_2 = 2+5$

$\Rightarrow \boxed{a_2 = 7}$

put a_1, a_2 value in equ (2) $\Rightarrow -5-7+a_4 = 3$

$\Rightarrow a_4 = 3+12$

$\Rightarrow \boxed{a_4 = 15}$

put a_1 value in equ (3) $\Rightarrow -5+a_3 = 4$

$\Rightarrow \boxed{a_3 = 9}$

$\therefore$ The co-ordinate matrix of $(2,3,4,-1)$ are $\begin{bmatrix} -5 \\ 7 \\ 9 \\ 15 \end{bmatrix}$

4)
i) Sol: Given $\alpha = (1,0,-1)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (0,1,-1)$

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$(4,5,6) = a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1)$$

$$(4,5,6) = (a_1 - a_2 + a_3,\ a_1 + a_2,\ a_1 + a_2 - a_3)$$

$$\therefore\ a_1 - a_2 + a_3 = 4 \rightarrow (1)$$

$$a_1 + a_2 = 5 \rightarrow (2)$$

$$a_1 + a_2 - a_3 = 6 \rightarrow (3)$$

solving (1) & (3)

$$a_1 - \cancel{a_2} + \cancel{a_3} = 4$$

$$a_1 + \cancel{a_2} - \cancel{a_3} = 6$$

$$2a_1 = 10$$

$$a_1 = 5$$

put a_1 in equ (2) $\rightarrow 5 + a_2 = 5$

$$\Rightarrow a_2 = 0$$

put a_1, a_2 in equ (3) $\Rightarrow 5 + 0 - a_3 = 6$

$$\cancel{\Rightarrow 10 - a_3 = 6} \Rightarrow a_3 = 6 - 5$$

$$\cancel{\Rightarrow -a_3 = -10 + 6} \Rightarrow a_3 = -1$$

$$\cancel{\Rightarrow a_3 = +4}$$

∴ The co-ordinate matrix of $(4,5,6)$ are $\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

Sol: Given $\alpha = (2,3,4,-1)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Where $\alpha_1 = (1,1,0,0)$

$\alpha_2 = (0,1,1,0)$

$\alpha_3 = (0,0,1,1)$

$\alpha_4 = (1,0,0,0)$.

let there exist scalars $a_1, a_2, a_3, a_4 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$(2,3,4,-1) = a_1(1,1,0,0) + a_2(0,1,1,0) + a_3(0,0,1,1) + a_4(1,0,0,0)$$

$$(2,3,4,-1) = (a_1 + a_4,\ a_1 + a_2,\ a_2 + a_3,\ a_3)$$

$$\therefore\ a_1 + a_4 = 2 \rightarrow (1)$$

$$a_1 + a_2 = 3 \rightarrow (2)$$

$$a_2 + a_3 = 4 \rightarrow (3)$$

$$a_3 = -1 \rightarrow (4)$$

$\alpha_2 = (1,1,0)$

$\alpha_3 = (1,0,2)$

let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$(1,0,-1) = a_1(0,1,-1) + a_2(1,1,0) + a_3(1,0,2)$$

$$(1,0,-1) = (a_2+a_3,\ a_1+a_2,\ -a_1+2a_3)$$

$\therefore$ $a_2 + a_3 = 1 \rightarrow (1)$

$a_1 + a_2 = 0 \rightarrow (2)$

$-a_1 + 2a_3 = -1 \rightarrow (3)$

Equ (2) & (3) $\Rightarrow$

solving (2) & (3)

$$\begin{array}{r} a_1 + a_2 = 0 \\ -a_1 + 2a_3 = -1 \\ \hline a_2 + 2a_3 = -1 \rightarrow (4) \end{array}$$

solving (1) & (4)

$$\begin{array}{r} a_2 + a_3 = 1 \\ a_2 + 2a_3 = -1 \\ \hline -a_3 = 2 \\ a_3 = 2 \end{array}$$

put a_3 in equ (1) $\Rightarrow a_2 - 2 = 1$

$a_2 = 3$

put a_2 in equ (2) $\Rightarrow a_1 + 3 = 0$

$\Rightarrow a_1 = -3$

$\therefore$ The co-ordinate matrix of $(1,0,-1)$ are $\begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}$

ii) sol Given $\alpha = (4,5,6)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,1,1)$

$\alpha_2 = (-1,1,1)$

$\alpha_3 = (1,0,-1)$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that